ADDENDUM TO OLIVIER SCHIFFMANN, "DRINFELD REALIZATION OF THE ELLIPTIC HALL ALGEBRA"

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Abstract. In [1] O. Schiffmann gave a presentation of the Drinfel'd double of the elliptic Hall algebra which is similar in spirit to Drinfel'd's new realization of quantum affine algebras. Using this result together with a part of his proof we can provide such a description for the elliptic Hall algebra.

We will use freely all the notations and the results of [1].
Let $\widetilde{\underline{\boldsymbol{\mathcal{E}}}}^+$ be the algebra generated by the Fourier coefficients of the series $\mathbb{T}_1(z)$ and $\mathbb{T}_0^+(z)$ subject only to the relevant positive relations (4.1), (4.2), (4.3), (4.5) in [1]. To avoid any confusion with the generators of $\widetilde{\boldsymbol{\mathcal{E}}}$ we denote the generators of $\widetilde{\underline{\boldsymbol{\mathcal{E}}}}^+$ by $\mathfrak{u}_{1,d}, d \in \mathbb{Z}$ and $\Theta_{0,d}, d \geq 1$.
We denote by $\widetilde{\boldsymbol{\mathcal{E}}}^{\pm}$ the **subalgebra** of $\widetilde{\boldsymbol{\mathcal{E}}}$ generated by the positive (resp. negative) generators. Similarly for $\boldsymbol{\mathcal{E}}^{\pm}$. Our goal is to prove that $\boldsymbol{\mathcal{E}}^+$ is isomorphic to $\widetilde{\underline{\boldsymbol{\mathcal{E}}}}^+$. The strategy is to go through their Drinfel'd doubles. But first we need to define a coalgebra structure on $\widetilde{\underline{\boldsymbol{\mathcal{E}}}}^+$.

Lemma 1.1. *The map* $\Delta : \widetilde{\underline{\boldsymbol{\mathcal{E}}}}^+ \to \widetilde{\underline{\boldsymbol{\mathcal{E}}}}^+ \widehat{\otimes} \widetilde{\underline{\boldsymbol{\mathcal{E}}}}^+$ *given on generators by*

$$\Delta(\mathbb{T}_0^+(z)) = \mathbb{T}_0^+(z) \otimes \mathbb{T}_0^+(z)$$

$$\Delta(\mathbb{T}_1(z)) = \mathbb{T}_1(z) \otimes 1 + \mathbb{T}_0^+(z) \otimes \mathbb{T}_1(z)$$

is a well defined algebra map and makes $\widetilde{\underline{\boldsymbol{\mathcal{E}}}}^+$ *into a (topological) bialgebra.*

Proof. We need to check that the map Δ respects all the relations between the generators of $\widetilde{\underline{\boldsymbol{\mathcal{E}}}}^+$. The relations (4.1), (4.2), (4.3) are an easy routine check. We are left to check the cubic relation (4.5). Using [1] Lemma 4.1 we only need to check the following relation:

$$[[\mathfrak{u}_{1,-1}, \mathfrak{u}_{1,1}], \mathfrak{u}_{1,0}] = 0.$$

Applying Δ we obtain:

$$[[\mathfrak{u}_{1,-1}, \mathfrak{u}_{1,1}], \mathfrak{u}_{1,0}] \otimes 1 + E + \sum_{m,n,l \geq 0} \Theta_{0,m}\Theta_{0,n}\Theta_{0,l} \otimes [[\mathfrak{u}_{1,-1-m}, \mathfrak{u}_{1,1-n}], \mathfrak{u}_{1,-l}] \tag{1.1}$$

where $E \in \widetilde{\underline{\boldsymbol{\mathcal{E}}}}^+[1] \widehat{\otimes} \widetilde{\underline{\boldsymbol{\mathcal{E}}}}^+[2] + \widetilde{\underline{\boldsymbol{\mathcal{E}}}}^+[2] \widehat{\otimes} \widetilde{\underline{\boldsymbol{\mathcal{E}}}}^+[1]$.
The first term is 0 since it's exactly the cubic relation. We want to prove that E and the third term are also 0. Let us begin with E.
We will need to use the following easy lemma whose proof is omitted:

Lemma 1.2. *Let A, B be two algebras over a field. Suppose we have a morphism of algebras $f : A \to B$. Then* $\ker(f \otimes f) = A \otimes \ker(f) + \ker(f) \otimes A$.

The arguments of [1] Section 5.3 show that $\widetilde{\underline{\boldsymbol{\mathcal{E}}}}^+[\leq 2]$ and $\boldsymbol{\mathcal{E}}^+[\leq 2]$ are isomorphic (through the canonical morphism). We apply the above lemma to this morphism

can : $\underline{\widetilde{\boldsymbol{\mathcal{E}}}}^+ \to \boldsymbol{\mathcal{E}}^+$ and we get in particular that

$$\underline{\widetilde{\boldsymbol{\mathcal{E}}}}^+[\leq 2] \otimes \underline{\widetilde{\boldsymbol{\mathcal{E}}}}^+[\leq 2] \to \boldsymbol{\mathcal{E}}^+[\leq 2] \otimes \boldsymbol{\mathcal{E}}^+[\leq 2]$$

is still an isomorphism.
Using the fact that the map **can** commutes with the coproduct we get that **can** $\otimes$ **can**$(E) = 0$. By the above isomorphism we deduce that $E = 0$.[1]
Let us now deal with the cubic term. For any integers $m, n, l \in \mathbb{Z}$ we put

$$R(m,n,l) = \sum_{(m,n,l)} [[\mathfrak{u}_{1,-1+m}, \mathfrak{u}_{1,1+n}], \mathfrak{u}_{1,l}]$$

where the sum is over all the six permutations of the triplet (m,n,l). So in order to prove that the third term of the relation (1.1) vanishes it is enough to prove that $R(m,n,l) = 0$ for any $m,n,l \in \mathbb{Z}$.
Observe first that $R(l,l,l) = 0$ for any $l \in \mathbb{Z}$ since it is the cubic relation (4.6) from [1]. By symmetry we can suppose that $l \leq m, n$. Applying the adjoint action of $\mathfrak{u}_{0,k-l}$ to the relation $R(l,l,l) = 0$ we get that $R(k,l,l) = 0$ for any $k \geq l$. So in particular $R(m,l,l) = 0$. Now applying the adjoint action of $\mathfrak{u}_{0,n-l}$ to $R(m,l,l) = 0$ we obtain $R(m,n,l) = 0$ which is exactly what we wanted. $\square$

In [1] it is proved that $\widetilde{\boldsymbol{\mathcal{E}}}^+$ is isomorphic to $\boldsymbol{\mathcal{E}}^+$. It follows that there is a natural surjective morphism $\pi : \underline{\widetilde{\boldsymbol{\mathcal{E}}}}^+ \to \widetilde{\boldsymbol{\mathcal{E}}}^+ \simeq \boldsymbol{\mathcal{E}}^+$ and therefore a natural surjective morphism on the Drinfel'd doubles:

$$D\underline{\widetilde{\boldsymbol{\mathcal{E}}}}^+ \to D\boldsymbol{\mathcal{E}}^+ \simeq \boldsymbol{\mathcal{E}} \simeq \widetilde{\boldsymbol{\mathcal{E}}}$$

If the natural map $\widetilde{\boldsymbol{\mathcal{E}}} \to D\underline{\widetilde{\boldsymbol{\mathcal{E}}}}^+$ is well defined then since the composition

$$\widetilde{\boldsymbol{\mathcal{E}}} \to D\underline{\widetilde{\boldsymbol{\mathcal{E}}}}^+ \to \widetilde{\boldsymbol{\mathcal{E}}}$$

is the identity (because all the morphisms are the obvious ones) we obtain that

$$\widetilde{\boldsymbol{\mathcal{E}}}^+ \simeq \underline{\widetilde{\boldsymbol{\mathcal{E}}}}^+$$

which is what we wanted.
To prove that the natural morphism $\widetilde{\boldsymbol{\mathcal{E}}} \to D\underline{\widetilde{\boldsymbol{\mathcal{E}}}}^+$ is well defined we need to check that the relations (4.1)-(4.5) are satisfied in $D\underline{\widetilde{\boldsymbol{\mathcal{E}}}}^+$. It is clear that (4.1), (4.3), (4.5) and (4.2) ($\epsilon_1 = \epsilon_2$) are satisfied since they involve only the positive (resp. negative) part at once. We need to deal with (4.2) ($\epsilon_1 = -\epsilon_2$) and (4.4). We claim that they are implied by Drinfel'd's relations in the double. This is an easy verification.
Putting all together we have:

Theorem 1.3. *The elliptic Hall algebra $\boldsymbol{\mathcal{E}}^+$ is isomorphic to the algebra generated by the Fourier coefficients of $\mathbb{T}_1(z)$ and $\mathbb{T}_0^+(z)$ subject to the relations:*

$$\mathbb{T}_0^+(z)\mathbb{T}_0^+(w) = \mathbb{T}_0^+(w)\mathbb{T}_0^+(z)$$
$$\chi_1(z,w)\mathbb{T}_0^+(z)\mathbb{T}_1(w) = \chi_{-1}(z,w)\mathbb{T}_1(w)\mathbb{T}_0^+(z)$$
$$\chi_1(z,w)\mathbb{T}_1(z)\mathbb{T}_1(w) = \chi_{-1}(z,w)\mathbb{T}_1(w)\mathbb{T}_1(z)$$
$$\mathrm{Res}_{z,y,w}[(zyw)^m(z+w)(y^2-zw)\mathbb{T}_1(z)\mathbb{T}_1(y)\mathbb{T}_1(w)] = 0, \ \forall m \in \mathbb{Z}$$

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[1]It looks like we cheated here because E lives only in a completion of the tensor product. However, each graded piece of E (remember that $\underline{\widetilde{\boldsymbol{\mathcal{E}}}}^+$ is $\mathbb{Z}^2$ graded) lives in an ordinary tensor product and hence we can apply the lemma.

References

[1] O. Schiffmann - *Drinfeld realization of Elliptic Hall Algebra*, to appear in Journal of Algebraic Combinatorics, (2011)

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